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Iterative Methods for Solving Split Feasibility Problem in Hilbert Space

A. Kılıçman *[1] and L.B. Mohammed [2]

[1,2] *Department of Mathematics, Faculty of Science, Universiti Putra Malaysia, 43400 Serdang Selangor, Malaysia*

E-mail: akilic@upm.edu.my, lawanbulama@gmail.com
* *Corresponding author*

ABSTRACT

Based on the recent work of Wang et al. (2012), in this paper, we construct a new algorithm for solving split feasibility problem for the class of total quasi-asymptotically nonexpansive and uniformly $\tau-$Lipschitzian mappings in Hilbert space and prove its strong convergence result. The result presented in this paper, not only extend the result of Wang et al. Wang et al. (2012), but also extend, improve and generalize several well-known results in the literature.

Keywords: Iterative Algorithm; Total Quasi-Asymptotically Nonexpansive; Uniformly $\tau-$ Lipschitzian; Split Feasibility Problem; Strong Convergence.

1. Introduction

Let W be a Banach space, K_1, K_2, be two Hilbert spaces, $\langle ., . \rangle$ be an inner product, $\|.\|$ stand for the corresponding norm, Δ and Ω be nonempty closed convex subset of K_1 and K_2 respectively, $M : K_1 \to K_2$ be a bounded linear operator and M^* be the adjoint of M. And also let I be the identity operator on K_1, $Fix(\Psi)$ to donate the fixed point set of Ψ i.e., $Fix(\Psi) = \{r \in K : \Psi r = r\}$, " $\to$ "and " $\rightharpoonup$ " to denote the weak and strong convergence, and $\omega_\omega(r_m)$ to denote the set of the cluster point of $\{r_m\}$ in the weak topology i.e., $\{$ there exists $\{r_{m_k}\}$ of $\{r_m\} \ni r_{m_k} \rightharpoonup r\}$.

The mapping $\Psi : K_1 \to K_1$ is said to be; quasi nonexpansive, if $Fix(\Psi) \neq \emptyset$ such that $\|\Psi w - z\| \leq \|w - z\|, \forall z \in Fix(\Psi)$ and $w \in K_1$, quasi-asymptotically nonexpansive, if $Fix(\Psi) \neq \emptyset$ and there exists a sequence $\{r_m\} \subseteq [1, \infty)$ with $r_m \to 1 \ni \forall m \geq 1$, $\|\Psi^m w - z\|^2 \leq r_m \|w - z\|^2$, $\forall z \in Fix(\Psi)$ and $w \in K_1$, total quasi-asymptotically nonexpansive, if $Fix(\Psi) \neq \emptyset$, and there exists nonnegative real sequences $\{v_m\}, \{\mu_m\}$ in $[0, \infty)$ with $v_m \to 0$ and $\mu_m \to 0$, and a strictly increasing continuous function $\delta : \Re^+ \to \Re^+$ with $\delta(0) = 0 \ni \forall\ m \geq 1$, $\|\Psi^m w - z\|^2 \leq \|w - z\|^2 + v_m \delta(\|w - z\|) + \mu_m, \forall z \in Fix(\Psi)$ and $w \in K_1$, and it is said to be contraction with the coefficient $\varphi \in (0, 1)$ if $\|\Psi(w) - \Psi(z)\| \leq \varphi \|w - z\|, \forall w, z \in K_1$, η-strongly monotone, if $\exists$ a constant $\eta > 0 \ni \langle \Psi w - \Psi z, w - z\rangle \geq \eta \|w - z\|, \forall w, z \in K_1$.

Remark: It's not difficult to see that, if $\Psi : K_1 \to K_1$ is a contraction mapping with coefficient $\varphi \in (0, 1)$, then $(I - \Psi)$ is $(1 - \varphi)-$strongly monotone, i.e.,

$$\langle (I - \Psi)w - (I - \Psi)z,\ w - z\rangle \geq (1 - \varphi) \|w - z\|^2, \forall w, z \in K_1. \tag{1}$$

A Banach space W is said to satisfy Opial's condition (see Opial (1967)) if for any sequence $\{r_m\} \subseteq W$ with $r_m \rightharpoonup r$ as $m \to \infty$, then

$$\liminf_{m\to\infty} \|r_m - r\| < \liminf_{m\to\infty} \|r_m - r^*\|, \quad \forall r^* \in W \text{ and } r^* \neq r.$$

It's well known that each Hilbert space satisfied the Opial's property.

And also Ψ is said to be; demiclosed at zero, if for any sequence $\{r_m\}$ in K_1, with

$$r_m \rightharpoonup r \text{ and } \Psi r_m \to 0 \text{ as } m \to \infty \Rightarrow \Psi r = 0,$$

uniformly $\tau-$Lipschitzian, if $\exists$ a constant $\tau > 0$ such that

$$\|\Psi^m w - \Psi^m z\| \leq \tau \|w - z\|, \forall w, z \in K_1,$$

and it's said to be semi-compact, if for any bounded sequence $r_m \subseteq K_1$ with $\lim_{m\to\infty} \|\Psi r_m - r_m\| = 0$, there exists sub-sequence $\{r_{m_k}\} \subseteq \{r_m\}$ such that $\{r_{m_k}\}$ converges strongly to some point $r^* \in K_1$.

The split feasibility problem (SFP) consist as find a vector r^* satisfying

$$r^* \in \Delta \ni Mr^* \in \Omega. \tag{2}$$

The SFP (2) has been intensively studied by numerous authors due to its various applications in many physical problems such as; in image restoration, computer tomography and radiation therapy treatment planning (see Censor et al. (2006, 2005, 2007)). Iterative algorithm for approximating fixed points of nonexpansive mapping, quasi-nonexpansive, quasi asymptotically nonexpansive, total quasi-asymptotically nonexpansive mapping and their generalizations which solves problem (2) have been studied by a number of authors for example sees Ansari and Rehan (2014), Byrne (2002), Mohammed and Kılıçman (2015), Wang et al. (2012), Xu (2006), Yang (2004), Zhao and Yang (2005) and the references therein. One of the popular method that solves problem (2) is the Byne's algorithm see (Censor et al. (2007)) whose generates a sequence $\{r_m\}$ by

$$r_{m+1} := \Psi_\Delta\Big(I + \sigma M^*\big(\Psi_\Omega - I\big)M\Big)r_m, \forall m \in \mathbb{N}, \tag{3}$$

where Ψ_Δ and Ψ_Ω are the orthogonal projection onto Δ and Ω respectively, M is a bounded linear mapping and M^* is the adjoint of M, and $\sigma \in (0, \frac{2}{L})$ with L being the spectral radius of the operator M^*M. Suppose that, problem (2) has a solution, it's not difficult to see that $r^* \in \Delta$ solves (2) if and only if it solves the following equation:

$$r^* = \Psi_\Delta\Big(I + \sigma M^*\big(\Psi_\Omega - I\big)M\Big)r^*, \forall r^* \in \Delta, \tag{4}$$

where $\sigma > 0$, Ψ_Δ, Ψ_Ω, M, and M^* as in (3) above. The Krasnosel'skii-Mann algorithm which is known as K-M algorithm see Krasnosel'skii (1955), Mann (1953), whose generate a sequence $\{r_m\}$ by

$$r_{m+1} := (1-\gamma_m)r_m + \gamma_m \Psi r_m, \forall m \geq 0, \tag{5}$$

where $\{\gamma_m\}$ is a sequence in $[0,1]$, $r_0 \in \Delta$ is chosen arbitrarily and Ψ is a nonexpansive mapping, that is $\|\Psi w - \Psi z\| \leq \|w - z\|$, $\forall w, z \in K$. It was proved in Reich (1979) that the sequence $\{r_m\}$ defined by (5) converged weakly to a common fixed point r^* of Ψ provided that $\{\gamma_m\}$ satisfies

$$\sum_{m=1}^{\infty}(1-\gamma_m)\gamma_m = \infty. \tag{6}$$

Algorithm (3) can be seen as a special case of algorithm (5), this is due to the fixed-point formulation of equation (4), one can apply algorithm (5) to the operator

$$\Psi_\Delta\Big(I + \sigma M^*\big(\Psi_\Omega - I\big)M\Big)$$

to obtain the following algorithm:

$$r_{m+1} :=(1-\gamma_m)r_m + \gamma_m\Psi_\Delta\Big(I + \sigma M^*\big(\Psi_\Omega - I\big)M\Big)r_m, \forall m \in \mathbb{N}, \tag{7}$$

where $\sigma \in (0, \frac{2}{L})$ and again L is the spectral radius of the operator M^*M. It's not difficult to see that as long as $\{\gamma_m\}$ satisfy equation (6), the sequence $\{r_m\}$ defined by algorithm (7) will converge weakly to the solution of problem (2). On the other hand, if problem (2) is not consistence, the Byne's algorithm converges to a minimizer of $\|\Psi_\Delta(Md) - (Md)\|$ over $d \in \Delta$, whenever such a minimizer exists.

Recently, Wang et al. (2012), introduced the following algorithm for solving SFP (2) whose generate a sequence $\{r_m\}$ by

$$\begin{cases} r_1 \in K, \\ u_m = r_m + \sigma M^*(\Phi^m - I)Mr_m, \\ r_{m+1} = (1-\beta_m)u_m + \beta_m \Psi^m u_m, \forall m \geq 1, \end{cases} \tag{8}$$

where $\sigma \in \left(0, \frac{1}{\|M\|^2}\right)$, $\beta_m \subset [0,1]$ satisfied the condition; $0 < \liminf\limits_{m\to\infty}\beta_m \leq \limsup\limits_{m\to\infty}\beta_m < 1$ and Φ, Ψ are total quasi-asymptotically nonexpansive and uniformly $\tau-$ Lipschitzian mappings satisfied some certain conditions see Wang et al. (2012). It was proved in Wang et al. (2012), the sequence $\{r_m\}$ defined by algorithm (8) converged weakly to the solution of SFP (2) and the strong convergence follows if Ψ is a semi-compact. This compactness type condition appear very strong as only few mappings are semi-compact. It's an interesting problem to continue studying this problem (SFP) and prove its strong convergence result without any compactness type condition assume.

It's the aim of this paper to modify the algorithm of Wang et al Wang et al. (2012) for the class of total quasi asymptotically nonexpansive and uniformly $\tau-$ Lipschitzian mappings so that the strong convergence is guaranteed for the solution of SFP (2).

In what follows, we denote the solution set of SFP (2) by Γ, i.e.,

$$\Gamma = \Big\{ r^* \in \Delta \text{ such that } Mr^* \in \Omega \Big\}. \tag{9}$$

2. Preliminaries

Lemma 2.1. *(Marino and Xu (2007)) Let K_1 be a real Hilbert space, then*

(i) $\|w+z\|^2 = \|w\|^2 + 2\langle w, z\rangle + \|z\|^2$, $\forall w, z \in K_1$

(ii) $\|kw + (1-k)z\|^2 = k\|w\|^2 + (1-k)\|z\|^2 - k(1-k)\|w-z\|^2$, $\forall w, z \in K_1$ *and* $k \in [0,1]$.

Lemma 2.2. *(Yang et al. (2011)) Let $\{r_m\} \subseteq K_1$ such that $r_m \rightharpoonup r$, then* $\limsup\limits_{m\to\infty} \|r_m - z\|^2 = \limsup\limits_{m\to\infty} \|r_m - r\|^2 + \|r - z\|^2, \forall z \in K_1$.

Lemma 2.3. *(Wang et al. (2012)) Let $\Psi : K_1 \to K_1$ be a $(\{v_m\}, \{\mu_m\}, \xi)$-total quasi-asymptotically nonexpansive mapping. Then for each $z \in Fix(\Psi)$, $w \in K_1$ and $m \geq 1$, the following inequalities are equivalent.*

(i) $\|\Psi^m w - z\|^2 \leq \|w - z\|^2 + v_m \xi(\|w - z\|) + \mu_m$,

(ii) $2\langle w - \Psi^m w, w - z\rangle \geq \|\Psi^m w - w\|^2 - v_m \xi(\|w - z\|) - \mu_m$,

(iii) $2\langle w - \Psi^m w, z - \Psi^m w\rangle \leq \|\Psi^m w - w\|^2 + v_m \xi(\|w - z\|) + \mu_m$.

Lemma 2.4. *(Yang et al. (2011)) Let C be a nonempty closed convex subset of K_1 and $\Psi : C \to C$ be a $(k, \{v_m\}, \{\mu_m\}, \xi)-$ total asymptotically strict pseudocontractive and uniformly $\tau-$Lipschitzian mapping. Then $I - \Psi$ is demiclosed at zero in the sense that if $\{r_m\}$ is a sequence in C such that $r_m \rightharpoonup r^*$ and $\limsup\limits_{m\to\infty} \|r_m - \Psi^m r_m\| = 0$, then $\Psi r^* = r^*$. In particular, if $r_m \rightharpoonup r^*$ and $(I - \Psi) r_m \to 0$ then $\Psi r^* = r^*$, i.e., Ψ is demiclosed at zero.*

Based on Lemma (2.4), we obtain the following lemma.

Lemma 2.5. *Let Δ be a nonempty closed convex subset of a Hilbert space K_1 and $\Psi : \Delta \to \Delta$ be a $(\{v_m\}, \{\mu_m\}, \xi)-$ total quasi-asymptotically nonexpansive mapping, then $I - \Psi$ is demiclosed at zero in the sense that, if $\{r_m\}$ is a sequence in Δ such that $r_m \rightharpoonup r^*$ and $(I - \Psi^m) r_m \to 0$, then $\Psi r^* = r^*$. In particular, if $r_m \rightharpoonup r^*$ and $(I - \Psi) r_m \to 0$ then $\Psi r^* = r^*$.*

Proof. By the boundedness of $\{r_m\}$, we can define a function g on K_1 by

$$g(r) = \limsup_{m\to\infty} \|r_m - r\|^2, \forall r \in K_1, \tag{10}$$

by Lemma (2.2) and the weak convergence of r_m, we have

$$g(r) = g(r^*) + \|r^* - r\|^2, \forall r \in K_1.$$

In particular, for $m \geq 1$,

$$g(\Psi^m r^*) = g(r^*) + \|r^* - \Psi^m r^*\|^2. \tag{11}$$

On the other hand, Ψ is $(\{v_m\},\{\mu_m\},\xi)-$ total quasi-asymptotically nonexpansive mapping, by (10), we get

$$\begin{aligned} g(\Psi^m r^*) &= \limsup_{m\to\infty} \|r_m - \Psi^m r^*\|^2 \\ &\leq \limsup_{m\to\infty}\Big(\|r_m - r^*\|^2 + v_m\xi(\|r_m - r^*\|) + \mu_m\Big) \\ &= g(r^*). \end{aligned} \tag{12}$$

By substituting (11) into (12), we have that

$$g(r^*) + \|r^* - \Psi^m r^*\|^2 \leq g(r^*),$$

which implies that $\Psi^m r^* = r^*$. Hence, $\Psi r^* = r^*$. □

Lemma 2.6. *(Tian and Di (2011)) Let $\{r_m\}$ and σ_m be two sequences of nonnegative real numbers satisfying*

$$r_{m+1} \leq (1-\delta_m)r_m + \sigma_m, m \geq 0,$$

where $\delta_m \subset (0,1)$ such that:

(i) $\lim\limits_{m\to\infty} \delta_m = 0$ and $\sum_{m=0}^{\infty} \delta_m = \infty$,

(ii) $\lim\limits_{m\to\infty} \frac{\sigma_m}{\delta_m} \leq 0$ or $\sum_{m=0}^{\infty} |\sigma_m| < \infty$, then the $\lim\limits_{m\to\infty} r_m = 0$.

Lemma 2.7. *(Wang et al. (2012)) Let $\{r_m\}$, $\{\delta_m\}$ and $\{\sigma_m\}$ be sequences of nonnegative real numbers satisfying*

$$r_{m+1} \leq (1+\delta_m)r_m + \sigma_m, m \geq 1,$$

if $\sum \delta_m < \infty$ and $\sum \sigma_m < \infty$, then the $\lim\limits_{m\to\infty} r_m$ exists.

Lemma 2.8. *(Bauschke and Borwein (1996)) If $\{r_m\}$ is a Fejer monotone with respect to Δ, then*

(i) $r_m \rightharpoonup r^* \in \Delta$ if and only if $\omega_\omega(r_m) \subset \Delta$;

(ii) The sequence $\{\Psi_\Delta r_m\}$ converges strongly to some point in Δ;

(iii) If $r_m \rightharpoonup r^* \in \Delta$, then $r^* = \lim\limits_{m\to\infty} \Psi_\Delta r_m$.

Lemma 2.9. *(Marino and Xu (2007)) Let Δ be a of K_1. Given $w \in K_1$ and $z \in \Delta$. Then $z = \Psi_\Delta w$ if and only if there hold the relation $\langle w - z, y - z\rangle \leq 0, \forall y \in \Delta$.*

3. Main Results

In this section, we present the main result of this paper which is the extension of the Theorem (3.1) of Wang et al. (2012).

Theorem 3.1. *Let K_1, K_2 be two Hilbert spaces, $\Psi : K_1 \to K_1$, $\Phi : K_2 \to K_2$ be $(\{v_{m_1}\}, \{\mu_{m_1}\}, \xi_1)$, $(\{v_{m_2}\}, \{\mu_{m_2}\}, \xi_2)-$ total quasi-asymptotically nonexpansive and uniformly $\tau_{1,2}-$Lipschitzian continuous mappings with $v_m = \max\{v_{m_1}, v_{m_2}\}$, $\mu_m = \max\{\mu_{m_1}, \mu_{m_2}\}$, $\xi = \max\{\xi_1, \xi_2\}$ and $\tau = \max\{\tau_1, \tau_2\}$ such that $\sum v_m < \infty$ and $\sum \mu_m < \infty$, and also let $h : K_1 \to K_1$ be a contraction mapping with the coefficient $\lambda \in (0, 1)$ and N, N^* be positive constants such that $\xi(\delta) \leq \xi(N) + N^*\delta^2, \forall \delta \geq 0$, $M : K_1 \to K_2$ be a bounded linear operator and $M^* : K_2 \to K_1$ be the adjoint of M with $L = \|MM^*\|$. Assume that $\Gamma \neq \emptyset$ and let $\{r_m\}$ be define by*

$$\begin{cases} r_0 \in K_1 \text{ is chosen arbitrary}, \\ u_m = r_m + \sigma M^*(\Phi^m - I)Mr_m, \\ y_m = (1-\beta)u_m + \beta\Psi^m(u_m), \\ r_{m+1} = \gamma_m h(r_m) + (1-\gamma_m)y_m, \forall m \geq 0, \end{cases} \tag{13}$$

where $\beta \in (0, 1)$, $\sigma \in (0, \frac{1}{L})$, and γ_m is sequence in $(0, 1)$ satisfy the conditions;

$$\begin{cases} (a) \ \lim_{m\to\infty} \gamma_m = 0 \text{ and } \sum_{m=0}^{\infty} \gamma_m = \infty, \\ (b) \ 0 < \eta < \gamma_m < 1. \end{cases} \tag{14}$$

Then the sequence $\{r_m\}$ defined by algorithm (13) converges to $r^* \in \Gamma$ which solves the variational inequality problem:

$$\langle (h - I)r^*, r - r^* \rangle \leq 0, \forall r \in \Gamma. \tag{15}$$

Note that, equation (15) is equivalent with $\Psi_\Gamma h(r^*) = r^*$ see Lemma (2.9), where Ψ_Γ is the metric projection of K_1 onto Γ.

Proof. **Step 1.** In this step, we show that $\{r_m\}$ is bounded.

Let $r^* \in \Gamma$, from (13) and Lemma (2.1), we have

$$\begin{aligned} \|r_{m+1} - r^*\|^2 &= \|\gamma_m h(r_m) + (1-\gamma_m)y_m - r^*\|^2 \\ &\leq \gamma_m \|h(r_m) - r^*\|^2 + (1-\gamma_m)\|y_m - r^*\|^2 \\ &\leq 2\gamma_m\lambda^2 \|r_m - r^*\|^2 + 2\gamma_m \|h(r^*) - r^*\|^2 + (1-\gamma_m)\|y_m - r^*\|^2. \end{aligned} \tag{16}$$

On the other hand,

$$\begin{aligned}
\|y_m - r^*\|^2 &= \|(1-\beta)u_m + \beta\Psi^m(u_m) - r^*\|^2 \\
&\leq (1-\beta)\|u_m - r^*\|^2 + \beta\|\Psi^m(u_m) - r^*\|^2 \\
&- \beta(1-\beta)\|\Psi^m(u_m) - u_m\|^2 \\
&\leq \|u_m - r^*\|^2 + \beta v_m \xi(\|u_m - r^*\|) \\
&- \beta(1-\beta)\|\Psi^m(u_m) - u_m\|^2 + \beta\mu_m \qquad (17)\\
&\leq (1+\beta v_m N^*)\|u_m - r^*\|^2 \\
&- \beta(1-\beta)\|\Psi^m(u_m) - u_m\|^2 + \beta\Big(v_m\xi(N) + \mu_m\Big), \qquad (18)
\end{aligned}$$

and

$$\begin{aligned}
\|u_m - r^*\|^2 &= \|r_m + \sigma M^*(\Phi^m - I)Mr_m - r^*\|^2 \\
&\leq \|r_m - r^*\|^2 + 2\sigma\langle Mr_m - Mr^*, (\Phi^m - I)Mr_m\rangle \\
&+ \sigma^2\langle MM^*(\Phi^m - I)Mr_m, (\Phi^m - I)Mr_m\rangle \\
&\leq \|r_m - r^*\|^2 + \sigma^2 L\|\Phi^m Mr_m - Mr_m\|^2 \\
&+ 2\sigma\langle \Phi^m Mr_m - Mr^* - (\Phi^m - I)Mr_m, (\Phi^m - I)Mr_m\rangle \\
&\leq \|r_m - r^*\|^2 - \sigma(2-\sigma L)\|\Phi^m Mr_m - Mr_m\|^2 \\
&+ 2\sigma\langle \Phi^m Mr_m - Mr^*, \Phi^m Mr_m - Mr_m\rangle. \qquad (19)
\end{aligned}$$

By Lemma (2.3), we deduce that

$$\begin{aligned}
2\sigma\langle \Phi^m Mr_m - Mr^*, \Phi^m Mr_m - Mr_m\rangle &\leq \sigma\|\Phi^m Mr_m - Mr_m\|^2 \\
&+ \sigma v_m\xi(\|Mr_m - Mr^*\|) + \sigma\mu_m \\
&\leq \sigma\|\Phi^m Mr_m - Mr_m\|^2 \\
+\sigma v_m N^*\|Mr_m - Mr^*\|^2 &+ \sigma\big(v_m\xi(N) + \mu_m\big). \qquad (20)
\end{aligned}$$

Substitute (20) into (19), we have

$$\begin{aligned}
\|u_m - r^*\|^2 &\leq \big(1 + \sigma v_m N^* L\big)\|r_m - r^*\|^2 \\
&- \sigma\big(1 - \sigma L\big)\|\Phi^m Mr_m - Mr_m\|^2 + \sigma\big(v_m\xi(N) + \mu_m\big). \qquad (21)
\end{aligned}$$

Substitute (21) into (18), we have

$$\begin{aligned}
\|y_m - r^*\|^2 &\leq (1+\beta v_m N^*)\big(1 + \sigma v_m N^* L\big)\|r_m - r^*\|^2 \\
&- (1+\beta v_m N^*)\sigma\big(1 - \sigma L\big)\|\Phi^m Mr_m - Mr_m\|^2 \\
&+ \Big(v_m\xi(N) + \mu_m\Big)\Big((1+\beta v_m N^*)\sigma + \beta\Big). \qquad (22)
\end{aligned}$$

Substitute (22) into (16), we have

$$\begin{aligned}
\|r_{m+1}-r^*\|^2 &\leq \Big(2\gamma_m\lambda^2 + (1-\gamma_m)(1+\beta v_m N^*)\big(1+\sigma v_m N^* L\big)\Big)\|r_m - r^*\|^2 \\
&\quad - (1-\gamma_m)(1+\beta v_m N^*)\sigma\big(1-\sigma L\big)\|\Phi^m M r_m - M r_m\|^2 \\
&\quad + (1-\gamma_m)\Big(v_m\xi(N)+\mu_m\Big)\Big((1+\beta v_m N^*)\sigma + \beta\Big) \\
&\quad + 2\gamma_m\|h(r^*)-r^*\|^2 \qquad (23)\\
&\leq \Big(1+\sigma v_m N^* L + 2\gamma_m\lambda^2 + \beta v_m N^*\big(1+\sigma v_m N^* L\big) \\
&\quad - \gamma_m(1+\beta v_m N^*)\big(1+\sigma v_m N^* L\big)\Big)\|r_m - r^*\|^2 \\
&+(1-\gamma_m)\Big(v_m\xi(N)+\mu_m\Big)\Big((1+\beta v_m N^*)\sigma+\beta\Big) + 2\gamma_m\|h(r^*)-r^*\|^2 \\
&\leq \Big(1+\sigma v_m N^* L + 2\lambda^2 + \beta v_m N^*\big(1+\sigma v_m N^* L\big)\Big)\|r_m - r^*\|^2 \\
&+(1-\eta)\Big(v_m\xi(N)+\mu_m\Big)\Big((1+\beta v_m N^*)\sigma+\beta\Big) + 2\|h(r^*)-r^*\|^2. \qquad (24)
\end{aligned}$$

It follows from (24) that

$$\|r_{m+1}-r^*\|^2 \leq \big(1+\delta_m\big)\|r_m - x^*\|^2 + \varphi_m, \quad \text{where} \tag{25}$$

$$\delta_m = \sigma v_m N^* L + 2\lambda^2 + \beta v_m N^*\big(1+\sigma v_m N^* L\big) \quad \text{and} \tag{26}$$

$$\varphi_m = (1-\eta)\Big(v_m\xi(N)+\mu_m\Big)\Big((1+\beta v_m N^*)\sigma+\beta\Big) + 2\|h(r^*)-r^*\|^2. \tag{27}$$

Evidently, from equation (26) and (27), we have

$$\sum_{m=1}^{\infty}\delta_m < \infty \text{ and } \sum_{m=1}^{\infty}\varphi_m < \infty. \tag{28}$$

By Lemma (2.7), we conclude that

$$\lim_{m\to\infty}\|r_m - r^*\| \text{ exists}, \tag{29}$$

therefore, $\{r_m\}$ is bounded. □

Step 2. In this step, we show that

$$\lim_{m\to\infty}\|r_m - r^*\| = \lim_{m\to\infty}\|u_m - r^*\| = \lim_{m\to\infty}\|y_m - r^*\|. \tag{30}$$

Proof. From (16), (22) and the fact that (29) exists, we have

$$\lim_{m\to\infty} \|r_m - r^*\| = \lim_{m\to\infty} \|y_m - r^*\|. \tag{31}$$

And also, from (18), (21) and the fact that (31) holds, we have

$$\lim_{m\to\infty} \|u_m - r^*\| = \lim_{m\to\infty} \|r_m - r^*\|. \tag{32}$$

Hence, equation (30) follows trivially from (31) and (32). □

Step 3. In this step, we show that

$$\lim_{m\to\infty} \|\Phi^m Mr_m - Mr_m\| = 0 \text{ and } \lim_{m\to\infty} \|\Psi^m u_m - u_m\| = 0. \tag{33}$$

Proof. The fact that Φ and Ψ are uniformly $\tau-$ Lipschitzian, and M, $\{r_m\}$, and $\{u_m\}$ are bounded, then $\{\Phi^m Mr_m\}$ and $\{\Psi^m u_m\}$ are also, and from (23), we have

$$\begin{aligned}
\sigma(1-\sigma L)\|\Phi^m Mr_m - Mr_m\|^2 &\leq \|r_m - r^*\|^2 - \|r_{m+1} - r^*\|^2 \\
&+ \Big[\big((1-\gamma_m)\beta v_m N^* - \gamma_m\big)(1+\sigma v_m M^* L) + 2\gamma_m\lambda^2 + \sigma v_m N^* L\Big] \\
&\times \|r_m - r^*\|^2 - \sigma(1-\sigma L)\big[(1-\gamma_m)\beta v_m N^* - \gamma_m\big]\|\Phi^m Mr_m - Mr_m\|^2 \\
+(1-\gamma_m)\Big(v_m\xi(N) + \mu_m\Big)&\Big((1+\beta v_m N^*)\sigma + \beta\Big) + 2\gamma_m\|h(r^*) - r^*\|^2
\end{aligned} \tag{34}$$

Evidently, from (29) and (34), we have

$$\lim_{m\to\infty} \|\Phi^m Mr_m - Mr_m\| = 0, \tag{35}$$

and from (18) and (30), we also have

$$\lim_{m\to\infty} \|\Psi^m u_m - u_m\| = 0. \tag{36}$$

Hence, equation (33) follows trivially from (35) and (36). □

Step 4. In this step, we show that

$$\lim_{m\to\infty} \|r_{m+1} - r_m\| = 0 \text{ and } \lim_{m\to\infty} \|u_{m+1} - u_m\| = 0. \tag{37}$$

Proof. From (13), we have

$$\begin{aligned}\|r_{m+1}-r_m\| &= \left\|\gamma_m\big(h(r_m)-y_m\big)+y_m-r_m\right\| \\ &\leq \gamma_m\,\|h(r_m)-y_m\|+\|u_m-r_m\|\,\beta\,\|\Psi^m u_m-u_m\| \\ &\leq \gamma_m\,\|h(r_m)-y_m\|+\beta\,\|\Psi^m u_m-u_m\| \\ &+\sigma\,\|M\|\,\|\Phi^m Mr_m-Mr_m\|\,. \end{aligned} \tag{38}$$

In view of (35) and (36) and the fact that $\{r_m\}$ and $\{y_m\}$ are bounded, we have

$$\lim_{m\to\infty}\|r_{m+1}-r_m\|=0. \tag{39}$$

Similarly, it follows from (35) and (39) that

$$\begin{aligned}\|u_{m+1}-u_m\| &\leq \|r_{m+1}-r_m\|+\|\sigma M^*(\Phi^{m+1}-I)Mr_{m+1}\| \\ &+\|\sigma M^*(\Phi^m-I)Mr_m\|\to 0 \text{ as } m\to\infty,\end{aligned}$$

$$\Rightarrow \lim_{m\to\infty}\|u_{m+1}-u_m\|=0. \tag{40}$$

Hence, equation (37) follows trivially from (39) and (40). □

Step 5. In this step, we show that

$$\|u_m-\Psi u_m\|\to 0 \quad \textit{and} \quad \|Mr_m-\Phi Mr_m\|\to 0 \text{ as } m\to\infty. \tag{41}$$

Proof. The fact that $\|u_m-\Psi^m u_m\|\to 0$, $\|u_{m+1}-u_m\|\to 0$ and Ψ is uniformly $\tau-$Lipschitzian continuous mapping, it follows that

$$\begin{aligned}\|u_m-\Psi u_m\| &\leq \|u_m-\Psi^m u_m\|+\|\Psi u_m-\Psi^m u_m\| \\ &\leq \|u_m-\Psi^m u_m\|+\tau\left\|u_m-\Psi^{m-1}u_m\right\| \\ &\leq \|u_m-\Psi^m u_m\|+\tau\left\|\Psi^{m-1}u_m-\Psi^{m-1}u_{m-1}\right\| \\ &+\tau\left\|u_m-\Psi^{m-1}u_{m-1}\right\| \\ &\leq \|u_m-\Psi^m u_m\|+\tau^2\,\|u_m-u_{m-1}\| \\ &+\tau\left\|u_m-u_{m-1}+u_{m-1}-\Psi^{m-1}u_{m-1}\right\| \\ &\leq \|u_m-\Psi^m u_m\|+\tau(\tau+1)\,\|u_m-u_{m-1}\| \\ &+\tau\left\|u_{m-1}-\Psi^{m-1}u_{m-1}\right\|\to 0 \\ &\Rightarrow \|u_m-\Psi u_m\|\to 0.\end{aligned}$$

Similarly, from the fact that $\|Mr_m-\Phi^m Mr_m\|\to 0$, $\|r_{m+1}-r_m\|\to 0$ and Φ is uniformly $\tau-$ Lipschitzian continuous mapping, it's not difficult to see that $\|Mr_m-\Phi Mr_m\|\to 0$. □

Step 6. In this step, we show that

$$r_m \rightharpoonup r^* \text{ and } u_m \rightharpoonup r^* \text{ as } m \to \infty. \tag{42}$$

In view of (30), we see that $\{r_m\}, \{u_m\}$ are bounded, then $\exists$ a sub-sequence u_{m_k} of u_m such that

$$u_{m_k} \rightharpoonup r^*, \text{ as } k \to \infty. \tag{43}$$

From (43) and (41), we deduce that

$$\|u_{m_k} - \Psi u_{m_k}\| \to 0, \text{ as } k \to \infty. \tag{44}$$

From (43), (44) and Lemma (2.5), we get that $r^* \in Fix(\Psi)$.

Moreover, from (13), (43) and the fact $\|Mr_m - \Phi^m Mr_m\| \to 0$, as $m \to \infty$, we have

$$r_{m_k} = u_{m_k} - \sigma M^*(\Phi^{m_k} - I)Mr_{m_k} \rightharpoonup r^*.$$

By the definition of M, we get

$$Mr_{m_k} \rightharpoonup Mr^* \text{ as } k \to \infty. \tag{45}$$

In view of (41), we get

$$\|Mr_{m_k} - \Phi Mr_{m_k}\| \to 0, \text{ as } k \to \infty. \tag{46}$$

From (45), (46) and Lemma (2.5), we have that $Mr^* \in Fix(\Phi)$, this implies that $r^* \in \Gamma$, that is r^* is a solution of SFP (2).

Now we prove (42).

Suppose by contradiction, there exists another u_{m_k} of u_m such that $u_{m_k} \rightharpoonup z^* \in \Gamma$ with $r^* \neq z^*$. By (30) and Opial's property, we have

$$\begin{aligned}
\liminf_{k\to\infty} \|u_{m_k} - r^*\| &< \liminf_{k\to\infty} \|u_{m_k} - z^*\| = \liminf_{m\to\infty} \|u_m - z^*\| \\
&= \liminf_{k\to\infty} \|u_{m_k} - z^*\| < \liminf_{k\to\infty} \|u_{m_k} - r^*\| \\
&= \liminf_{m\to\infty} \|u_m - r^*\| \\
&= \liminf_{k\to\infty} \|u_{m_k} - r^*\| \\
\Rightarrow \liminf_{k\to\infty} \|u_{m_k} - r^*\| &< \liminf_{k\to\infty} \|u_{m_k} - r^*\|
\end{aligned}$$

which is contradiction. Therefore, $u_m \rightharpoonup r^*$. By using (13) and (33) , we have $r_m = u_m - \sigma M^*(\Phi^m - I)Mr_m \rightharpoonup r^*$ as $m \to \infty$. Thus, by Lemma (2.8) we deduce that

$$\Omega_\Omega(r_n) \subset \Gamma. \tag{47}$$

Step 7. In this step, we show that

$$\limsup_{m\to\infty} \langle (h-I)r^*, r_m - r^* \rangle \leq 0. \tag{48}$$

Indeed from (**??**), there exists a sub-sequence $\{r_{m_k}\} \subset \{r_m\}$ such that

$$\begin{aligned}\limsup_{m\to\infty} \langle (h-I)r^*, r_m - r^* \rangle &= \limsup_{k\to\infty} \langle (h-I)r^*, r_{m_k} - r^* \rangle \\ &\leq 0. \end{aligned} \tag{49}$$

Suppose without loss of generality that $r_{m_k} \rightharpoonup r$, from (**??**), it follows that $r \in \Gamma$. Since r^* is the unique solution of (15), implies that

$$\begin{aligned}\limsup_{m\to\infty} \langle (h-I)r^*, r_m - r^* \rangle &= \limsup_{k\to\infty} \langle (h-I)r^*, r_{m_k} - r^* \rangle . \\ &= \langle (h-I)r^*, r - r^* \rangle \leq 0. \end{aligned} \tag{50}$$

Step 8. Finally, we show that

$$r_m \to r^* \text{ as } m \to \infty. \tag{51}$$

Proof.

$$\begin{aligned}
\|r_{m+1} - r^*\|^2 &= \langle \gamma_m h(r_m) + (1-\gamma_m) y_m - r^*, r_{m+1} - r^* \rangle \\
&= \gamma_m \langle h(r_m) - r^*, r_{m+1} - r^* \rangle + (1-\gamma_m) \langle y_m - r^*, r_{m+1} - r^* \rangle \\
&\leq \gamma_m \langle h(r_m) - h(r^*), r_{m+1} - r^* \rangle + \gamma_m \langle h(r^*) - r^*, r_{m+1} - x^* \rangle \\
&+ \frac{(1-\gamma_m)}{2} \|y_m - r^*\|^2 + \frac{(1-\gamma_m)}{2} \|r_{m+1} - r^*\|^2 \\
&\leq \frac{\gamma_m \lambda^2}{2} \|r_m - x^*\|^2 + \frac{\gamma_m}{2} \|r_{m+1} - r^*\|^2 + \frac{(1-\gamma_m)}{2} \|y_m - r^*\|^2 \\
&+ \frac{(1-\gamma_m)}{2} \|r_{m+1} - r^*\|^2 + \gamma_m \langle h(r^*) - r^*, r_{m+1} - r^* \rangle \\
&= \frac{(1-\gamma_m)}{2} \|y_m - r^*\|^2 + \frac{\gamma_m \lambda^2}{2} \|r_m - r^*\|^2 + \frac{1}{2} \|r_{m+1} - r^*\|^2 \\
&+ \gamma_m \langle h(r^*) - r^*, r_{m+1} - r^* \rangle \\
\Rightarrow \|r_{m+1} - r^*\|^2 &\leq \big(1-\gamma_m\big) \|y_m - r^*\|^2 + \gamma_m \lambda^2 \|r_m - r^*\|^2 \\
&+ 2\gamma_m \langle h(r^*) - r^*, r_{m+1} - r^* \rangle .
\end{aligned} \tag{52}$$

From (17), we deduced that

$$\|y_m - r^*\|^2 \leq \|u_m - r^*\|^2 + \beta v_m \xi(\|u_m - r^*\|) + \beta \mu_m. \tag{53}$$

In view of (19) and (20), we deduced that

$$\|u_m - r^*\|^2 \leq \|r_m - r^*\|^2 + \sigma v_m \xi(\|Mr_m - Mr^*\|) + \sigma \mu_m. \tag{54}$$

Substituting (54) into (53), we have

$$\begin{aligned} \|y_m - r^*\|^2 \leq \|r_m - r^*\|^2 &+ \sigma v_m \xi(\|Mr_m - Mr^*\|) \\ &+ \beta v_m \xi(\|u_m - r^*\|) + \mu_m(\beta + \sigma). \end{aligned} \tag{55}$$

Substituting (55) into (52), we have

$$\begin{aligned} \|r_{m+1} - r^*\|^2 \leq \big(1 - \gamma_m(1-\lambda^2)\big) &\|r_m - r^*\|^2 \\ &+ \big(1 - \gamma_m\big)\sigma v_m \xi(\|Mr_m - Mr^*\|) \\ &+ \big(1 - \gamma_m\big)\beta v_m \xi(\|u_m - r^*\|) \\ &+ \big(1 - \gamma_m\big)\mu_m(\beta + \sigma) \\ &+ 2\gamma_m \left\langle h(r^*) - r^*, r_{m+1} - r^* \right\rangle. \end{aligned} \tag{56}$$

It follows from (56) that

$$\|r_{m+1} - r^*\|^2 \leq \big(1 - \varphi_m\big) \|r_m - r^*\|^2 + \rho_m, \quad \text{where} \tag{57}$$

$$\varphi_m = \gamma_m(1 - \lambda^2), \tag{58}$$

and

$$\begin{aligned} \rho_m = \big(1 - \gamma_m\big)\sigma v_m \xi(\|Mr_m - Mr^*\|) + \big(1 - \gamma_m\big)\beta v_m \xi(\|u_m - r^*\|) \\ + \big(1 - \gamma_m\big)\mu_m(\beta + \sigma) + 2\gamma_m \left\langle h(r^*) - r^*, r_{m+1} - r^* \right\rangle. \end{aligned} \tag{59}$$

It follows From (58) and (59) that

$$\lim_{m\to\infty} \varphi_m = 0 \text{ and } \sum_{m=1}^{\infty} \varphi_m = \infty. \tag{60}$$

In view of (48), (58) and (59), we have

$$\begin{aligned} \lim_{m\to\infty} \frac{\rho_m}{\varphi_m} &\leq \frac{(1-\eta)}{\eta(1-\lambda^2)} \Big(\sigma v_m \xi(\|Mr_m - Mr^*\|) \beta v_m \xi(\|u_m - r^*\|) + \mu_m(\beta + \sigma)\Big) \\ &+ \frac{2}{(1-\lambda^2)} \left\langle h(r^*) - r^*, r_{m+1} - r^* \right\rangle \\ &= 0. \end{aligned} \tag{61}$$

Hence, by Lemma (2.6) we conclude that

$r_m \to r^*$ as $m \to \infty$. □

Taking $\gamma_m = 0$ and $\beta = \beta_m$ in Theorem (3.1), we get

Corollary 3.1. *(see Wang et al. (2012)) Let K_1, K_2 be two Hilbert spaces, $\Psi : K_1 \to K_1$, $\Phi : K_2 \to K_2$ be two uniformly $\tau-$Lipschitzian and $(\{v_m\}, \{\mu_m\}, \xi)-$ total quasi-asymptotically nonexpansive mappings with $\sum v_m < \infty$ and $\sum \mu_m < \infty$ such that Ψ and Φ are demiclosed at zero. Let N and N^* be positive constants such that $\xi(\delta) \leq \xi(N) + N^*\delta^2, \forall \delta \geq 0$, and let $M : K_1 \to K_2$ be a bounded linear operator and M^* be the adjoint of M with $L = \|MM^*\|$. Assume that $\Gamma \neq \emptyset$, and let $\{r_m\}$ be the sequence generated by (8), where $\beta_m \subseteq [0, 1]$ such that $0 < \liminf_{m\to\infty} \beta_m \leq \limsup_{m\to\infty} \beta_m < 1$, and $\sigma \in (0, \frac{1}{L})$. Then the sequence $\{r_m\}$ defined by (8) converges weakly to $r^* \in \Gamma$.*

Corollary 3.2. *Let K_1, K_2 be two Hilbert spaces, $\Psi : K_1 \to K_1$, $\Phi : K_2 \to K_2$ be $\{v_{m_1}\}$, $\{v_{m_2}\}-$ quasi-asymptotically nonexpansive mappings with $v_m = \max\{v_{m_1}, v_{m_2}\}$such that $\sum v_m < \infty$ and let M, M^*, L, h, β, σ, $\{\gamma_m\}$ and $\{r_m\}$ be as in Theorem (3.1). Assume that $\Gamma \neq \emptyset$,. Then $\{r_m\}$ converges strongly to $r^* \in \Gamma$ which solves the VIP (15).*

Proof. Ψ and Φ are $(\{v_m\}, \{\mu_m\}, \xi)-$ total quasi-asymptotically nonexpansive and uniformly $\tau-$ Lipschitzian mappings with $\{v_m\} = \{k_m - 1\}, \mu_m = 0$, $\xi(\delta) = \delta^2, \forall \delta \geq 0$ and $\tau = \sup_{m\geq 1} v_m$ respectively. Therefore, all the conditions in Theorem (3.1) are satisfied . Hence, the conclusions of this Corollary follows directly from Theorem (3.1). □

Corollary 3.3. *Let K_1, K_2 be two Hilbert spaces, $\Psi : K_1 \to K_1$, $T : K_2 \to K_2$ be two quasi-nonexpansive mappings, and let M, M^*, L, h, β, σ, $\{\gamma_m\}$ and $\{r_m\}$ be as in Theorem (3.1). Assume that $\Gamma \neq \emptyset$,. Then $\{r_m\}$ converges strongly to $r^* \in \Gamma$ which solves the VIP (15).*

Proof. Ψ and Φ are $(\{1\})-$ quasi-asymptotically nonexpansive and uniformly $(\{1\})-$ Lipschitzian mappings. Therefore, all the conditions in Corollary (3.2) are satisfied. Hence, the conclusions of this Corollary follows directly from Corollary (3.2). □

4. Conclusion

The results presented in this paper is the extension of the result of Wang et al. (2012) in the sense that;

- Take $\gamma_m = 0$ and $\beta = \beta_m$ in Theorem (3.1), then our algorithm reduces to Wang et al. (2012) algorithm (8).
- The Wang's et al Wang et al. (2012) result gave a weak convergence result for the solution of SFP (9) and the strong convergence follows only if Ψ is a semi-compact, while our result gave the strong convergence result for the solution of SFP (9) without imposing the condition that Ψ is a semi-compact.
- In Wang et al. (2012) result, the demicloseness of $I - \Psi$ and $I - \Phi$ was imposed, while in our result the demicloseness of $I - \Psi$ and $I - \Phi$ was proved see Lemma (2.5).

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